# Outlook

## §1 $\mathbb{Q}\otimes AV$ is semi-simple

$k$ field

<u>Def</u> $X/k$ simple $\underset{\text{def}}{=}$ $\nexists\ 0 \neq Y \subsetneq X$

<u>Rmk</u> Can happen $X$ simple, $\bar{k} \underset{k}{\otimes} X$ not simple.

<u>Poincaré reducibility</u> $Y \subseteq X$ AVs

Then $\exists\ Z \subseteq X$ AV s.th.

$$Y \times Z \longrightarrow X \quad \text{is an isogeny.}$$

<u>Equivalent</u>: $\dim Z = \dim X - \dim Y$

& $Z \cap Y$ finite.

<u>Cor</u> (by induction)

Given $X$, $\exists\ X_1, \ldots, X_r$ simple s.th.

$$X \sim X_1^{e_1} \times \cdots \times X_r^{e_r} \qquad (\star)$$

$\sim \underset{\text{def}}{=} \exists$ isogeny.

<u>Lem</u> Let $\varphi : X_1 \longrightarrow X_2$ map of simple AVs. Then $\varphi = 0$ or $\varphi$ is an isogeny

<u>Proof</u> $\tilde{Y}_1 = \ker \varphi \qquad \tilde{Y}_2 = \varphi(X_1)$

are group schemes.

Then seen last time that

$$X_i' := (\tilde{Y}_i^{\,0})_{red} \text{ is an AV.}$$

If $Y_1 = X_1$, then $\varphi = 0$.

If $Y_1 = 0$, then $\varphi$ finite kernel

and $Y_2 \neq 0$ (if $X_1 \neq 0$)

hence $Y_2 = X_2$. $\Longrightarrow \varphi$ isogeny. $\blacksquare$

Consequence 1  If $X_i \not\sim X_j$ for $i \neq j$,

then $X_i$, $e_i$ uniquely determined up to isogeny in $(*)$ and reordering.

Then $\mathrm{End}^0(X) = \prod_{i=1}^{r} M_{e_i}(\mathrm{End}^0(X_i))$

Consequence 2  $\mathrm{End}^0(X_i)$ are skew-fields.

Equivalent: Any $0 \neq \varphi : X_i \to X_i$ is an isogeny.

Converse  If $\mathrm{End}^0(X)$ is skew-field, then $X$ is simple.

Our only example of AVs so far is

$$X \sim E_1^{e_1} \times \cdots \times E_r^{e_r} \qquad g = e_1 + \cdots + e_r$$

## §2 CM - abelian varieties

We seek AV $X/\mathbb{C}$ with interesting $\mathrm{End}^0(X)$.

(cf. Chai - Conrad - Oort)

<u>Idea</u> $K/\mathbb{Q}$ deg $2g$

<u>Recall</u>: Archimedean Place $K$

$\overline{\overline{\text{def}}}$ $K \xhookrightarrow{\sigma} \mathbb{C}$ up to conjugation $\sigma \longmapsto \overline{\sigma}$.

$\sigma$ <u>real</u> if $\sigma(K) \subseteq \mathbb{R}$

$\sigma$ <u>complex</u> if $\sigma(K) \nsubseteq \mathbb{R}$.

Assume all archimedean places of $K$ to be complex, i.e.

$\mathrm{Hom}(K, \mathbb{C}) = \{\sigma_1, \overline{\sigma_1}, \dots, \sigma_g, \overline{\sigma_g}\}$

<u>Def</u> <u>CM type</u> $\Phi \subseteq \mathrm{Hom}(K, \mathbb{C})$

s.th. $\#\Phi = g$ , $\Phi \cap \overline{\Phi} = \emptyset$.

i.e. choice of $\{\sigma_i, \overline{\sigma_i}\}$ $\forall i$

one out of

View $\Phi \doteq (\varphi_1, \dots, \varphi_g) : K \longrightarrow \mathbb{C}^g$

Then $X = \mathbb{C}^g / \Phi(\mathcal{O}_K)$ is complex

torus. Since $\Lambda = \Phi(\mathcal{O}_K)$ (a lattice)

is stable under mult by $K$,

$\Longrightarrow K \hookrightarrow \mathrm{End}^\circ(X)$

<u>Crucial</u> $X$ need not be an AV

(i.e. need not exist $X \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ )

<u>Def</u> $K$ <u>CM-field</u> if $\exists \gamma \in \mathrm{Aut}(K/\mathbb{Q})$

s.th. $\gamma^2 = \mathrm{id}$ & $\sigma \circ \gamma = \overline{\sigma}$

for all $\sigma : K \hookrightarrow \mathbb{C}$.

<u>Fact</u> If $K$ CM-field, then

$X$ is an AV.

<u>Reason</u> $K \times K \to \mathbb{Q}$

$$\langle x, y \rangle = \mathrm{tr}_{K|\mathbb{Q}}(\delta \cdot x \cdot \gamma(y))$$

$\delta \in K^\times$ purely imaginary i.e. $\gamma(\delta) = -\delta$

is an <u>alternating</u> perfect pairing.

$$\langle x, x \rangle = \mathrm{tr}_{K|\mathbb{Q}}(\delta \cdot x \gamma(x))$$

$$= \sum_{\sigma : K \to \mathbb{C}} \sigma(\delta x (\gamma(x)))$$

$$= \sum_{\{\sigma, \overline{\sigma}\}} |\sigma(x)|^2 \cdot (\sigma(\delta) - \sigma(\delta))$$

$$= 0.$$

<u>Fact</u> Ample line bundle on $\mathbb{C}^g/\Lambda$

$\longleftrightarrow$ Riemann forms on $\Lambda$ :

$E : \Lambda \times \Lambda \to \mathbb{Z}$ alternating, non-degenerate

$+ \; E(ix, iy) = E(x, y)$

$+$ positivity property. (Mumford §1-3)

Then above pairing on $K \times K$ allows to define ample lb on $X$.

<u>Example</u> $K_0/\mathbb{Q}$ imag-quad (is CM-field)

$F/\mathbb{Q}$ totally real

$$\left( \mathbb{R} \otimes_{\mathbb{Q}} F \cong \mathbb{R} \times \cdots \times \mathbb{R} \right)$$

Then $K_0 \cdot F$ is CM-field.

(CM-involution comes from $K_0$.)

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Now assume $X \sim E_1^{e_1} \times \dots \times E_r^{e_r}$.

Then $K \hookrightarrow M_{e_i}(\mathrm{End}^\circ(E_i))\ \forall i$.

<u>Note</u> $K \hookrightarrow M_e(F)$

$\longleftrightarrow$ $K$-vsp structure on $F^e$.

in prtic $[K:\mathbb{Q}] \mid e \cdot [F:\mathbb{Q}]$.

$\Rightarrow$ In our case,

$r = 1$, $X \sim E^g$, $E$ has CM by $K_0$, say.

$K \hookrightarrow M_g(K_0)$

<u>Claim</u> $K_0 \subseteq K$.

<u>Proof</u> $K_0$ = center of $M_g(K_0)$, so commutes w/ $K$.

If $K_0 \not\subseteq K$, then composite $L = K_0 \cdot K$ is field w/ $[L : \mathbb{Q}] = 4g$.

Cannot act on $K_0^g$ $\lightning$.

<u>Conclusion</u> If $K$ does not contain imag-quad field, $X \not\sim$ (product of ECs)

## <u>§3 Moduli of AVs</u> $\quad \mathcal{H} = \mathbb{C} \setminus \mathbb{R}$

<u>Recall</u>

$$\begin{tikzcd}
\{ECs/\mathbb{C} + \mathbb{Z}^2 \xrightarrow{\cong} \pi_1(E,0)\}\cancel{/\cong} \arrow[rr, "\cong", "\beta"'] \arrow[d, "proj"] & & \mathcal{H} \arrow[d, "proj"] \\
\{ECs/\mathbb{C}\}/\cong \arrow[rr, "\cong"] & & GL_2(\mathbb{Z})\backslash\mathcal{H}
\end{tikzcd}$$

Most common generalization:

Torus $X = \mathbb{C}^g/\Lambda$ is AV

$\iff \exists$ Riemann form $\Lambda \times \Lambda \xrightarrow{E} \mathbb{Z}$

For the ample line bundle $L$ constructed from $E$,

$$\deg(\phi_L : X \longrightarrow X^\vee) = |\det E|$$

One gets

$\{$ g-dim AVs $X$ + principal polarization + isomorphism $= \phi_L$ some ample $L$

$\lambda : X \xrightarrow{\simeq} X^\vee \} / \cong$

$\cong \{ \Lambda \subseteq \mathbb{C}^g$ + perfect Riemann form $E \} / \cong$

i.e. $E : \Lambda \xrightarrow{\simeq} \Lambda^\vee$

$\Longleftrightarrow \det E = \pm 1$.

Fact Every $2g$-dimensional lattice $\Lambda$ + perfect alternating $E$ is

$$\cong \left( \mathbb{Z}^g \oplus \mathbb{Z}^g , \begin{pmatrix} & 1_g \\ -1_g & \end{pmatrix} \right)$$

( i.e. Class number of $Sp_{2g}$ is $= 1$ )

std basis.

Consider now $(X, \lambda, e_1, \dots, e_g, f_1, \dots, f_g)$

$\quad$ g-dim AV $\quad$ princ. pol $\quad \pi_1(X, 0)$)

Recall $\beta$: $(E, (e_1, f_1))$

$\longmapsto \quad e_1^{-1} \cdot f_1 \in \mathcal{H}.$

Analogue $\quad e_1, \dots, e_g \in \operatorname{Lie} X$

form a $\mathbb{C}$-basis

In these coordinates, $(f_1, \dots, f_g) = \Omega$

$\in Gl_{g \times g}(\mathbb{C}).$

Elliptic curves diagram generalizes:

$$\begin{tikzcd}
\left\{ (X,\lambda) + \text{std basis } e_1, \ldots, f_g \right\}/\cong \arrow[r, "\cong"] \arrow[d, "\text{proj}"] & \mathcal{H}_g \arrow[d, "\text{proj}"] \\
\left\{ (X,\lambda) \right\}/\cong \arrow[r, "\cong"] & Sp_{2g}(\mathbb{Z}) \backslash \mathcal{H}_g
\end{tikzcd}$$

$\longmapsto \Omega$

$$\mathcal{H}_g = \left\{ X + iY \in \mathrm{Sym}_{g\times g}(\mathbb{C}) \;\middle|\; Y \text{ is pos defn.} \right\}$$

<u>Def</u> $\mathcal{H}_g$ <u>Siegel half space</u>

$$\mathcal{A}_g := Sp_{2g}(\mathbb{Z}) \backslash \mathcal{H}_g$$

<u>Siegel modular variety.</u>

<u>Observation</u> $\dim_{\mathbb{C}} \mathcal{A}_g = \dim_{\mathbb{C}} \mathcal{H}_g$

$$= \frac{g(g+1)}{2}$$

How to think about

$$\mathcal{D} = \{ (X,\lambda) \in \mathcal{A}_g \mid X \sim \prod X_i^{e_i} \text{ non-trivially} \}$$

Fixing $g_i = \dim X_i$, $e_i \geq 1$

s.th. $\sum e_i g_i = g$, get

$$\mathcal{A}_{g_1} \times \cdots \times \mathcal{A}_{g_r} \longrightarrow \mathcal{A}_g$$

$$(X_1, \lambda_1) \times \cdots \times (X_r, \lambda_r) \longmapsto \prod_i (X_i, \lambda_i)^{e_i}.$$

$\mathcal{D}$ is then countable union of images of similar maps.

Since $\sum \frac{g_i(g_i+1)}{2} < \frac{g(g+1)}{2}$

in this case,

every generic AV is simple.

Similar argument works for endomorphism rings: Generically, $\mathrm{End}^0(X) = \mathbb{Q}$.

## §4 Next time

1) Thm Mordell-Weil $K/\mathbb{Q}$ finite, $X/K$ AV. Then

$$X(K) \text{ has finite rank.}$$

Motivation for lecture:

Birch-Swinnerton-Dyer Conj $E/\mathbb{Q}$ EC.

$$\mathrm{rk}_{\mathbb{Z}} E(\mathbb{Q}) = \mathrm{ord}_{s=1} L(s, E/\mathbb{Q})$$

Gross-Zagier obtained:

If RHS $= 1$, then the LHS is $\geq 1$.

i.e. If RHS = 1, they construct a non-torsion point in $E(\mathbb{Q})$

Idea $\forall N \geq 1$, $\exists X_0(N)/\mathbb{Q}$

"modular curve of level $\Gamma_0(N)$"

$$
\begin{tikzcd}
X_0(N)(\mathbb{C}) = \ker\left(GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/N)\right)\backslash\mathcal{H} \arrow[d, "\text{covering}"'] \\
SL_2(\mathbb{Z})\backslash\mathcal{H}
\end{tikzcd}
$$

Thm ((Taylor-) Wiles)

$\forall E/\mathbb{Q}$, $\exists N$ + non-constant

$$X_0(N) \xrightarrow{\alpha} E.$$

$$\left( \Longleftrightarrow E \xrightarrow{\neq 0} \mathrm{Pic}^0_{X_0(N)/\mathbb{Q}} \right)$$

Construct a ~~\$~~
point as image $\alpha(P)$
for Heegner point $P \in X_0(N)$.

( $P$ = $\Sigma$ ECs w/ complex multiplication. )

Gross-Zagier Formula

Implies that $\alpha(P)$ not torsion

if $\operatorname{ord}_{s=1} L(s, E/\mathbb{Q}) = 1$.

2) Construct $X_0(N)$ as a ~~cur~~
curve $/\mathbb{Q}$, even as a scheme

$$X_0(N) \longrightarrow \operatorname{Spec} \mathbb{Z}$$

3) Algebraic Geometry of GZ-Formula.

Modular forms will occur:

$$\{ E/\mathbb{Q} \text{ up to isogeny} \} \xrightarrow[\sim]{1:1} \left\{ \begin{array}{l} \text{Newforms of weight } 2 \\ \text{with } q\text{-expansion} \in \mathbb{Q}[[q]] \end{array} \right\}$$

Reformulation of Taylor-Wiles.

$$L(s, E/\mathbb{Q}) = L(s, f).$$